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ON THE NONEXISTENCE OF GLOBALLY BOUNDED SOLUTIONS TO INITIAL-HI--ETC(U)
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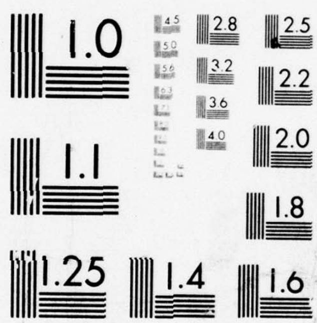

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(18) AFOSR (19) TR- 79-0831	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
TITLE (6) On the Nonexistence of Globally Bounded Solutions to Initial-History Value Problems for Integrodifferential Equations.		5. TYPE OF REPORT & PERIOD COVERED (9) Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
AUTHOR(s) (10) Prof. Frederick Bloom	(15)	8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3396
PERFORMING ORGANIZATION NAME AND ADDRESS University of South Carolina Math, Computer Science, and Statistics Columbia, S.C. 29208		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F (16) 2304 A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research NM Bolling, AFB D.C. 20332		12. REPORT DATE 6/18/79 (17) A4
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (11) 18 Jun 79		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; distribution unlimited (12) 20p.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Submitted to: Applicable Analysis

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Integrodifferential Equation; Maxwell-Hopkinson Dielectric.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Conditions are given under which it is impossible that there exist a solution $u \in C^2([0,\infty); (H_0^1(\Omega))^3)$ of the integrodifferential initial-history value problem — next page

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

$$\begin{cases} \underset{\sim}{u}_{tt} - \underset{\sim}{N}\,\underset{\sim}{u} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \underset{\sim}{0} \\ \underset{\sim}{u}(0) = \underset{\sim}{u}_0,\ \underset{\sim}{u}_t(0) = \underset{\sim}{v}_0\ (\underset{\sim}{u}_0, \underset{\sim}{v}_0 \in (H_0^1(\Omega))^3) \\ \underset{\sim}{u}(\tau) = \underset{\sim}{U}(\tau),\ -\infty < \tau < 0 \end{cases}$$

which lies in a class of bounded perturbations of the form

$$N_\infty = \{\underset{\sim}{v} \in C^2([0,\infty);\ (H_0^1(\Omega))^3) \mid \sup_{[0,\infty)} \|\underset{\sim}{v}\|_{(H_0^1(\Omega))^3} \le N\}$$

for some $N > 0$, where $\underset{\sim}{N}$, $\underset{\sim}{K}(t) \in L_S((H_0^1(\Omega))^3;\ (H^{-1}(\Omega))^3)$, $t \in (-\infty,\infty)$. An application is given to the nonexistence of globally bounded solutions, to initial-history boundary value problems for Maxwell-Hopkinson dielectrics, which lie in classes of bounded functions of the form N_∞.

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AFOSR-TR- 79-0831

LEVEL II

2 B.S.

On the Nonexistence of Globally Bounded Solutions*

to Initial-History Value Problems for Integrodifferential Equations

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*Research supported in part by AFOSR Grant 77-3396

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1. Introduction

In several recent papers [1] - [4] we have reported results on growth estimates for solutions to initial-history boundary value problems associated with integrodifferential equations and have presented applications in mechanics (viscoelasticity) and electromagnetic theory (rigid, nonconducting material dielectrics). The problems examined may all be placed in the following common abstract setting: Let H, H_+ be real Hilbert spaces with inner-products $< , >$ and $< , >_+$ respectively, and associated natural norms $||(\cdot)||$ and $||(\cdot)||_+$; we assume that $H_+ \subseteq H$ both algebraically and topologically with $\gamma > 0$ the embedding constant for the identity map i: $H_+ \to H$. Let H_- be the dual space of H_+ via the inner product of H, so that

$$||\underset{\sim}{v}||_- = \sup_{\underset{\sim}{w} \in H_+} [| < \underset{\sim}{v}, \underset{\sim}{w} > |/||\underset{\sim}{w}||_+], \text{ and let}$$

$$\underset{\sim}{N} \in L_s(H_+,H_-),\ \underset{\sim}{K}(\cdot) \in L_2((-\infty,\infty); L_s(H_+,H_-))$$

where $L_s(H_+,H_-)$ denotes the space of all bounded linear maps from H_+ into H_-. We then consider the abstract initial-history value problem for $\underset{\sim}{u} \in C^2([0,T); H_+)$, $T > 0$

$$(1.1) \quad \begin{cases} \underset{\sim}{u}_{tt} - \underline{Nu} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underline{u}(\tau)d\tau = \underline{0} \\ \underset{\sim}{u}(0) = \underline{u}_0,\ \underline{u}_t(0) = \underline{v}_0\ (\underline{u}_0, \underline{v}_0 \in H_+) \\ \underset{\sim}{u}(\tau) = \underline{U}(\tau),\ -\infty < \tau < 0 \end{cases}$$

Different assumptions on the past history $\underset{\sim}{U}$ are made in the various applications, i.e. in [2] it is assumed that $\underline{U}(\cdot) \in C^1((-\infty,0); H_+)$ with

$\lim_{t\to 0^-} ||\underline{U}(t) - \underline{f}||_+ = 0$, $\lim_{t\to 0^-} ||\underline{U}_t - \underline{g}||_+ = 0$, $\lim_{t\to -\infty} ||\underline{U}(t)||_+ = 0$ and $\int_{-\infty}^{0} ||\underline{U}(\tau)||_+ d\tau < \infty$ while in [3] we assume that $\underline{U} = \underline{0}$, $-\infty < t < 0$. In each of the applications, however, no definiteness assumptions are made relative to $\underline{N}$ or $\underline{K}(\cdot)$. The derivations in [1] - [4] are based on a logarithmic convexity argument which requires that we restrict our attention, a priori, to a class of bounded perturbations; this idea of stabilizing an otherwise unstable solution of the ill-posed problem (1.1) is due to F. John [5] and for the system (1.1) it is shown in [1] - [4] that the appropriate classes of bounded perturbations are of the form

(1.2) $$\mathcal{N} = \{\underset{\sim}{v} \in C([0,T); H_+) \mid \sup_{[0,T)} ||\underline{v}||_+ \le N\}$$

for some $N > 0$. In fact for solutions $\underset{\sim}{u} \in C^2([0,T); H_+)$ of (1.1) which lie in a class of bounded perturbations of the form $\mathcal{N}$, it has been shown [2] that the real-valued function

(1.3) $$F(t;\beta,t_0) = ||\underline{u}(t)||^2 + \beta(t + t_0)^2, \quad 0 \le t < T,$$

with β, t_0 arbitrary nonnegative real numbers, satisfies the differential inequality

(1.4) $$FF'' - F'^2 \ge -2F(2\mathcal{F}(0) + \beta), \quad 0 \le t < T$$

where

(1.5) $$\mathcal{F}(t) = \mathcal{E}(t) + k_1 \sup_{[0,\infty)} ||\underline{K}(t)||_{\mathcal{L}_s(H_+,H_-)} + k_2 \sup_{[0,\infty)} ||\underline{K}_t(t)||_{\mathcal{L}_s(H_+,H_-)}$$

with $\mathcal{E}(t) = \frac{1}{2}||\underline{u}_t||^2 - \frac{1}{2} < \underline{u}, \underline{N}\underline{u} >$ and k_1, k_2 computable nonnegative constants. Besides the obvious assumptions on $\underline{K}$ required by $\mathcal{F}(0) < \infty$ the only other assumption made relative to either $\underset{\sim}{N}$ or $\underset{\sim}{K}(\cdot)$ is that

$$(1.6)\qquad \begin{cases} -\langle \underline{v}, \underline{K}(0)\underline{v}\rangle \ge \kappa \|\underline{v}\|_+^2, & \underline{v} \in H_+ \\ \underline{\text{with}} & \\ \kappa \ge T\gamma \sup\limits_{[0,\infty)} \|\underline{K}_t(t)\|_{L_s(H_+,H_-)}. & \end{cases}$$

In the application to one-dimensional isothermal viscoelasticity (1.6) reduces to the statement that $g'(0) \le -\kappa$, $\kappa \ge T \sup\limits_{[0,\infty)} |g''(t)||$ where $g(\cdot)$ is the relaxation function of the material; it is well-known that the relaxation functions of one-dimensional isothermal viscoelastic materials must be monotonically decreasing in time [6] and (1.6) simply says that $g(\cdot)$ must be decreasing sufficiently fast at $t = 0$.

Growth estimates for $\|\underline{u}\|^2$ follow directly from the differential inequality (1.4) once special assumptions regarding $E(0)$, β, t_0, and the initial data $\underline{u}_0$, $\underline{v}_0$ are made; these estimates apply, as well, to the well-posed situations previously considered in the literature, i.e., [7]. The restriction in [1] - [4] to time intervals of the form $[0,T)$, $T < \infty$ and solutions $\underline{u} \in \mathcal{N}$ is crucial for the logarithmic convexity argument. The purpose of this note is to prove that under certain well-defined circumstances it is impossible that there exists a solution $\underline{u} \in C^2([0,\infty); H_+)$ of (1.1) which is globally bounded in the sense that it belongs to a class of bounded perturbations of the form

$$(1.7)\qquad \mathcal{N}_\infty = \{\underline{v} \in C([0,\infty); H_+) \mid \sup_{[0,\infty)} \|\underline{v}\|_+ \le N\}$$

for a prescribed $N > Q$; this will be accomplished by employing a mixed logarithmic convexity - concavity argument of the type previously used by this author in [6];

an application to the evolution of the electric induction field in a class of rigid nonconducting material dielectrics is presented in §3.

2. The Global Nonexistence Theorem

In anticipation of the application we have in mind we shall assume a past history $\underset{\sim}{U}$ of the form

$$\underline{U}(t) = \begin{cases} \underline{0}, & -\infty < t < -t_h \\ \underline{U}_h(t), & -t_h \le t < 0 \end{cases} \tag{2.1}$$

where $t_h > 0$ is an arbitrary real number, and $||\underline{U}_h||_+ \in L_2[-t_h,0]$. The hypothesis (1.6) may be weakened to

$$- < \underline{v}, \underline{K}(0)\underline{v} > \ \ge 0, \ \forall \underline{v} \in H_+ \tag{2.2}$$

and we also assume that

$$\begin{cases} K(t) = ||\underline{K}(t)||_{L_s(H_+,H_-)} \text{ satisfies } K(\cdot) \in L_1[0,\infty) \\ K^*(t) = \int ||\underline{K}_t||_{L_s(H_+,H_-)} dt \text{ satisfies} \\ K^*(\cdot) \in L_1[0,\infty) \text{ with } K^*(0) = 0. \end{cases} \tag{2.3}$$

The class N_∞ may be modified, in view of (2.1) to

$$N'_\infty = \{\underline{v} \in C([-t_h,\infty); H_+) \mid \sup_{[-t_h,\infty)} ||\underline{v}||_+ \le N\}$$

for N > 0 and finite and we then want to show that the following result obtains:

Theorem Suppose $K(\cdot)$ satisfies (2.2) and (2.3) and $E(0) = \frac{1}{2}||\underline{v}_0||^2 - \frac{1}{2} < \underline{u}_0, \underline{N}\,\underline{u}_0 > \ < 0$ with $< \underline{u}_0, \underline{v}_0 > \ > 0$. If, for $N_0 > 0$ (finite)

$$(2.4)\qquad |E(0)| \geq \frac{3\gamma}{2} N_0^2(||K||_{L_1[0,\infty)} + ||K^*||_{L_1[0,\infty)})$$

there can not exist a strong[(1)] solution of (1.1) which lies in $N'_{0,\infty}$ (N'_∞ with $N = N_0$).

<u>Proof</u>: Assume that $\underset{\sim}{u}$ is a strong solution of (1.1) with $\underline{u} \in N'_{0,\infty}$ i.e., $\sup_{[-t_h,\infty)} ||\underline{u}||_+ \leq N_0$, and set $G(t) = ||\underset{\sim}{u}(t)||^2$. Then $G' = 2 < \underset{\sim}{u}, \underset{\sim}{u}_t >$, $G'' = 2||\underline{u}_t||^2 + 2 < \underline{u}, \underline{u}_t >$ and for any $\beta > 0$ we have by direct computation

$$(2.5)\qquad GG'' - (\beta+1)G'^2 = 4(\beta+1)\mathcal{U}_\beta^2 + 2G(<\underline{u}, \underline{u}_{tt}> - (2\beta+1)||\underline{u}_t||^2)$$

where $\mathcal{U}_\beta^2(t) = ||\underline{u}||^2||\underline{u}_t||^2 - < \underline{u}, \underline{u}_t > \geq 0$ by the Schwarz inequality. Thus for any $\beta > 0$

$$(2.6)\qquad GG'' - (\beta+1)G'^2 \geq 2GR_\beta,\ 0 \leq t < \infty$$

where, in view of (1.1) and (2.1)

$$(2.7)\qquad R_\beta(t) = < \underline{u}, \underline{N}\,\underline{u} > - (2\beta+1)||\underline{u}_t||^2 - < \underline{u}, \int_{-t_h}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau >$$

Now rewrite (2.7) in the equivalent form

$$\begin{aligned}(2.8)\qquad R_\beta(t) &= -(2\beta+1)(||\underline{u}_t||^2 - < \underline{u}, N\,\underline{u} >) \\ &\quad - 2\beta < \underline{u}, \underline{N}\,\underline{u} > - < \underset{\sim}{u}, \int_{-t_h}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau > \\ &= -2(2\beta+1)E(t) - 2\beta < \underline{u}, \underline{N}\,\underline{u} > \\ &\quad - < \underline{u}, \int_{-t_h}^t \underline{K}(t-\tau)\underline{u}(\tau)d\tau >\end{aligned}$$

(1) $\underline{u} \in C^2([0,\infty); H_+)$ is a strong solution provided $\underline{u}_t \in C^1([0,\infty); H_+)$ and $\underline{u}_{tt} \in C([0,\infty); H_-)$.

and then take the inner-product (in H) of (1.1) with $\underset{\sim}{u}_t$ and integrate to obtain

$$(2.9) \qquad E(t) = E(0) - \int_0^t < \underset{\sim}{u}_\tau, \int_{-t_h}^{\tau} \underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}(\lambda)d\lambda > d\tau$$

Sustitution of (2.9) into (2.8) now yields

$$(2.10) \qquad R_\beta(t) = -2(2\beta+1)E(0) - 2\beta < \underset{\sim}{u}, \underset{\sim}{N}\,\underset{\sim}{u} >$$

$$+ 2(2\beta+1) \int_0^t < \underset{\sim}{u}_\tau, \int_{-t_h}^{\tau} \underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}(\lambda)d\lambda > d\tau$$

$$- < \underset{\sim}{u}, \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau >$$

We now take the inner-product (in H) of (1.1) with $\underset{\sim}{u}(t)$ and obtain, in view of the definition of $G(t)$

$$(2.11) \qquad \tfrac{1}{2}G'' = ||\underset{\sim}{u}_t||^2 + < \underset{\sim}{u}, \underset{\sim}{N}\,\underset{\sim}{u} > - < \underset{\sim}{u}, \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau >$$

which, in turn, implies that

$$(2.12) \qquad -2\beta < \underset{\sim}{u}, \underset{\sim}{N}\,\underset{\sim}{u} > = -\beta F'' + 2\beta||\underset{\sim}{u}_t||^2$$

$$-2\beta < \underset{\sim}{u}, \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau >$$

Substituting from (2.12) into (2.10) then yields

$$(2.13) \qquad R_\beta(t) = -\beta G'' - 2(2\beta+1)E(0) - (2\beta+1) < \underset{\sim}{u}, \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau >$$

$$+ 2(2\beta+1) \int_0^t < \underset{\sim}{u}_\tau, \int_{-t_h}^{\tau} \underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}(\lambda)d\lambda > d\tau$$

and, therefore, the differential inequality (2.6) is equivalent to

$$(2.14) \qquad GG'' - \left(\frac{\beta+1}{2\beta+1}\right) G'^2 \geq -4GE(0) - 2G < \underline{u}, \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau >$$

$$+ 4G \int_0^t < \underline{u}_\tau, \int_{-t_h}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda > d\tau$$

$$= 2G[2|E(0)| - < \underline{u}, \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau >$$

$$+ 2 \int_0^t < \underline{u}_\tau, \int_{-t}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda > d\tau].$$

However,

$$< \underline{u}_\tau, \int_{-t_h}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda > = - < \underline{u}(\tau), \underline{K}(0)\underline{u}(\tau) >$$

$$- < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda >$$

$$+ \frac{d}{d\tau} < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}(\tau-\lambda)\underline{u}(\lambda)d\lambda >$$

and, therefore, (2.14) may be rewritten in the form

$$(2.15) \qquad GG'' - \left(\frac{\beta+1}{2\beta+1}\right) G'^2 \geq 2G[2|E(0)| -$$

$$2 \int_0^t < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda >$$

$$- 2 < \underline{u}_0, \int_{-t_h}^{0} \underline{K}(-\tau)\underline{u}(\tau)d\tau >$$

$$+ < \underline{u}, \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau >]$$

where, in view of our hypothesis relative to $\underline{K}(0)$ we have dropped the term $-\int_0^t < \underline{u}(\tau), \underline{K}(0)\underline{u}(\tau) > d\tau$. The following estimates for the integrals on the right-hand side of (2.15) now obtain:

$$| < \underline{u}_0, \int_{-t_h}^{0} \underline{K}(-\tau)\underline{u}(\tau)d\tau > | \le$$

$$||\underline{u}_0|| \int_{-t_h}^{0} ||\underline{K}(-\tau)||_{L_s(H_+,H_-)} ||\underline{u}(\tau)||_+ d\tau \le$$

$$\gamma||\underline{u}_0||_+ \sup_{[-t_h,0]} ||\underline{u}||_+ \int_{-t_h}^{0} ||\underline{K}(-\tau)||_{L_s(H_+,H_-)} d\tau \le$$

$$\gamma\left(\sup_{[-t_h,0]} ||\underline{u}||_+\right)^2 \int_0^{t_h} ||\underline{K}(\tau)||_{L_s(H_+,H_-)} d\tau \le$$

$$\gamma N_0^2 ||K||_{L_1[0,\infty)}$$

so that

$$(2.16) \qquad - < \underline{u}_0, \int_{-t_h}^{0} \underline{K}(-\tau)\underline{u}(\tau)d\tau > \ \ge -\gamma N_0^2 ||K||_{L_1[0,\infty)}$$

Also,

$$| < \underline{u}, \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau > | \le$$

$$||\underline{u}|| \int_{-t_h}^{t} ||\underline{K}(t-\tau)||_{L_s(H_+,H_-)} ||\underline{u}(\tau)||_+ d\tau \le$$

$$\gamma\left(\sup_{[-t_h,\infty)} ||\underline{u}||_+\right)^2 \int_{-t_h}^{t} ||\underline{K}(t-\tau)||_{L_s(H_+,H_-)} d\tau$$

$$\le \gamma N^2 \int_0^{t+t_h} ||\underline{K}(\lambda)||_{L_s(H_+,H_-)} d\lambda$$

$$\le \gamma N^2 ||K||_{L_1[0,\infty)}$$

and, therefore,

$$(2.17) \qquad < \underline{u}, \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{u}(\tau)d\tau > \ \ge -\gamma N_0^2 ||K||_{L_1[0,\infty)}$$

Finally,

$$\left| \int_0^t < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda > d\tau \right| \le$$

$$\int_0^t \left| < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda > \right| d\tau \le$$

$$\int_0^\infty (||\underline{u}(\tau)|| \int_{-t_h}^{\tau} ||\underline{K}_\tau(\tau-\lambda)||_{L_s(H_+,H_-)} ||\underline{u}(\lambda)||_+ d\lambda) d\tau$$

$$\le \gamma \left(\sup_{[-t_h,\infty)} ||\underline{u}||_+\right)^2 \int_0^\infty \int_{-t_h}^{\tau} ||\underline{K}_\tau(\tau-\lambda)||_{L_s(H_+,H_-)} d\lambda d\tau$$

$$= \gamma \left(\sup_{[-t\ ,\infty)} ||\underline{u}||_+\right)^2 \int_0^\infty \int_0^{\tau+t_h} ||\underline{K}_\rho(\rho)||_{L_s(H_+,H_-)} d\rho d\tau$$

$$\le \gamma N_0^2 \int_0^\infty (K^*(\rho) \Big|_0^{\tau+t_h}) d\tau$$

$$= \gamma N_0^2 \int_0^\infty K^*(\tau+t_h) d\tau$$

$$= \gamma N_0^2 \int_{t_h}^\infty K^*(\lambda) d\lambda \le \gamma N^2 ||K^*||_{L_1[0,\infty)}$$

in view of our assumption that $K^*(0) = 0$. Thus,

$$- \int_0^t < \underline{u}(\tau), \int_{-t_h}^{\tau} \underline{K}_\tau(\tau-\lambda)\underline{u}(\lambda)d\lambda > d\tau \ge -\gamma N_0^2 ||K^*||_{L_1[0,\infty)} \tag{2.18}$$

Combining the estimates (2.16) - (2.18) with the differential inequality (2.15) now yields

$$GG'' - \left(\frac{\beta+1}{2\beta+1}\right)G'^2 \ge 2G[2|E(0)| - 3\gamma N_0^2\{||K||_{L_1[0,\infty)} + ||K^*||_{L_1[0,\infty)}\}] \tag{2.19}$$

and in view of our hypothesis (2.4), relative to $|E(0)|$, (2.19) implies that

$$GG'' - \left(\frac{\beta+1}{2\beta+1}\right)G'^2 \ge 0, \quad 0 \le t < \infty. \tag{2.20}$$

Now set $\alpha(\beta) = \beta+1/2\beta+1$; note that $\frac{1}{2} < \alpha(\beta) < 1$, for all $\beta > 0$, with $\alpha(\beta) \to \frac{1}{2}^+$ as $\beta \to +\infty$ and $\alpha(\beta) \to 1^-$ as $\beta \to 0^+$. Let $H(t)$ be any real valued nonnegative twice continuously differentiable function on $[0,\infty)$ and γ any real number $0 < \gamma < 1$; then

$$(2.21) \qquad [H^{(1-\gamma)}]''(t) = (1-\gamma)H^{-\gamma-1}(t)[H(t)H''(t) - \gamma H'^2(t)]$$

Applying this last identity with $G(t)$ in place of $H(t)$, and $\gamma = \alpha(\beta)$, for any $\beta > 0$, and using (2.20) we easily find that

$$(2.22) \qquad [G^{(1-\alpha(\beta))}]''(t) \geq 0, \quad 0 \leq t < \infty$$

which implies that

$$(2.23) \qquad [G^{(1-\alpha(\beta))}]'(t) \geq [G^{(1-\alpha(\beta))}]'(0)$$

$$= (1-\alpha(\beta))G^{-\alpha(\beta)}(0)\, G'(0)$$

A second integration then yields

$$(2.24) \qquad G^{(1-\alpha(\beta))}(t) \geq (1-\alpha(\beta))G^{-\alpha(\beta)}(0)G'(0)t + G^{(1-\alpha(\beta))}(0)$$

$$= G^{(1-\alpha(\beta))}(0)[1 + (1-\alpha(\beta))\left(\frac{G'(0)}{G(0)}\right)t]$$

and as $(1-\alpha(\beta)) > 0$, for all $\beta > 0$,

$$(2.25) \qquad G(t) \geq G(0)\,[1 + (1-\alpha(\beta))\left(\frac{G'(0)}{G(0)}\right)t]^{\frac{1}{1-\alpha(\beta)}}$$

Clearly, it follows from (2.25) and the fact that $G'(0) = 2 < \underline{u}_0, \underline{v}_0 > > 0$

that $\lim_{t \to +\infty} G(t) = +\infty$. But $G(t) = ||\underline{u}(t)||^2$ and therefore $\lim_{t \to +\infty} ||\underline{u}(t)||^2 = +\infty$.

However, via the embedding of H_+ into H, $||\underline{u}(t)|| \leq \gamma ||\underline{u}(t)||_+$, $0 \leq t < \infty$, and so $\lim_{t \to +\infty} ||\underline{u}(t)||_+ = +\infty$, which implies that $\sup_{[-t_h,\infty)} ||\underline{u}(t)||_+ = +\infty$, contradicting the assumption that $\underline{u} \in N'_{0,\infty}$

Q.E.D.

Remark The above Theorem has, of course, an immediate converse, i.e., if there exists a strong solution $\underline{u} \in N'_{0,\infty}$ of (1.1) then $E(0)$ must either be nonnegative or, if $E(0) < 0$, then $|E(0)|$ must be sufficiently small, i.e.,

$$|E(0)| < \frac{3}{2}\gamma N_0^2(||K||_{L_1[0,\infty)} + ||K^*||_{L_1[0,\infty)})$$

If $\underset{\sim}{N}$ satisfies an appropriate definiteness condition, e.g., $< \underline{v}, \underline{N}\,\underline{v} > \geq \lambda ||\underline{v}||_+^2$, $\forall \underline{v} \in H_+$, $\lambda > 0$ then

$$2E(0) = ||\underline{v}_0||^2 - < \underline{u}_0, \underline{Nu}_0 >$$

$$\leq ||\underline{v}_0||^2 - \lambda ||\underline{u}_0||_+^2 < 0$$

Provided $\lambda ||\underline{u}_0||_+^2 > ||\underline{v}_0||^2$. In particular we would then have

$2|E(0)| > \lambda ||\underline{u}_0||_+^2 - ||\underline{v}_0||^2$ and, therefore, if

$$||\underline{u}_0||_+^2 \geq \frac{1}{\lambda} ||\underline{v}_0||^2 + \frac{3\gamma}{\lambda} (||K||_{L_1[0,\infty)} + ||K^*||_{L_1[0,\infty)}) N_0^2$$

$$\equiv \frac{1}{\lambda} ||\underline{v}_0||^2 + X(\gamma,\lambda,N_0)$$

it follows that no strong solution of (1.1), with initial data $\underline{u}_0$, $\underline{v}_0$, could exist in $N'_{0,\infty}$; this result is non-trivial only if

$$X(\gamma,\lambda,N_0) < N_0^2$$

which will certainly be the case, for any $N_0 > 0$ and finite, if the coerciveness constant λ is sufficiently large and/or the embedding constant γ is sufficiently small.

Remark Under appropriate assumptions on $E(0)$ relative to

$\sup_{[0,\infty)} ||\underline{K}||_{L_s(H_+,H_-)}$ and $\sup_{[0,\infty)} ||\underline{K}_t||_{L_s(H_+,H_-)}$ it has been shown in [2] that $||\underline{u}||$ must grow exponentially on $[0,T)$, $T > 0$ finite, for $\underline{u} \in N$. If we replace N by $N'_{0,\infty}$ the exponential growth estimates for $||\underset{\sim}{u}||$ in [2] remain valid on $[0,T)$ but we can not take the limit as $T \to +\infty$ in [2] in view of the basic hypothesis of that paper, i.e., (1.6).

Remark The estimate (2.25), valid for all $\beta > 0$, in fact implies that

$$G(t) \geq G(0) \lim_{\beta \to 0^+} [1 + (1-\alpha(\beta))\left(\frac{G'(0)}{G(0)}\right)t]^{\frac{1}{1-\alpha(\beta)}}$$

$$= G(0) \exp\left(\frac{G'(0)}{G(0)} t\right), \quad 0 \leq t < \infty$$

in view of $\alpha(\beta) \to 1^-$ as $\beta \to 0^+$ and the elementary fact that $\lim_{\lambda \to 0^+} (1+\lambda x)^{\frac{1}{\lambda}} = e^x$.

3. An Application in Electromagnetic Theory

In two recent works [3], [7], we have considered the behavior of electric fields and electric displacement fields in rigid, nonconducting dielectrics with memory of the type introduced by Hopkinson [8] in an effort to understand the phenomena of residual charge in Leyden jars. The Maxwell-Hopkinson dielectric is governed by the pair of constitutive equations

$$(3.1) \quad \begin{cases} \underline{D}(\underline{x},t) = \epsilon\, \underline{E}(\underline{x},t) + \int_{-\infty}^{t} \phi(t-\tau)\underline{E}(\underline{x},\tau)d\tau \\ \\ \underline{H} = \mu^{-1}\, \underline{B} \end{cases}$$

where, of course, $\underline{D}$, $\underline{E}$, $\underline{H}$, and $\underline{B}$ are , respectively, the electric displacement (or induction) field, the electric field, the magnetic intensity, and the magnetic induction. It is usually assumed that $\phi(t)$, $t \geq 0$, is a (sufficiently) smooth and monotonically decreasing function of t. The fields $\underline{D}$, $\underline{H}$ are introduced so as to simplify the formulation of Maxwell's equations and, in particular, $\underline{D} \equiv \epsilon_0 \underline{E} + \underline{P}$ where $\epsilon_0 > 0$ is a known physical constant and $\underline{P}$ is the polarization field vector in the dielectric, which is assumed to be nonconducting (i.e., no free mobile charges so that div $\underline{D} = 0$).

Let $\Omega \subseteq R^n$ be a bounded open domain with smooth boundary $\partial\Omega$ (smooth enough to apply the divergence theorem). Let $\tilde{\Omega} \subset \Omega$. We assume that $\Omega/\tilde{\Omega}$ is filled with a perfect conductor, so that $\underline{D} = \underline{0}$ in $\Omega/\tilde{\Omega}$ and on $\partial\Omega$, and that $\tilde{\Omega}$ is filled with a nonconducting rigid dielectric of Maxwell-Hopkinson type so that the constitutive equations (3.1) apply in $\tilde{\Omega}$. On $\partial\tilde{\Omega}$, i.e., at the interface between the dielectric and the perfect conductor, $\underline{D} \cdot \underline{n} = 0$ where $\underset{\sim}{n}$ is the unit outward normal to $\partial\tilde{\Omega}$. It has been shown in [3] that if we combine Maxwell's equations in $\tilde{\Omega}$, with

(i) the constitutive relations (3.1) with

$$\underline{E}(\underset{\sim}{x},\tau) = \begin{cases} \underline{0}, & -\infty < \tau < -t_h \\ \underline{E}_h(\underset{\sim}{x},\tau), & -t_h \leq \tau < 0 \end{cases}$$

for some $t_h > 0$

(ii) the inverted constitutive equation

$$\underline{E}(\underset{\sim}{x},t) = \epsilon^{-1}\underline{D}(\underset{\sim}{x},t) + \epsilon^{-1} \int_{-t_h}^{t} \Phi(t-\tau)\underline{D}(\underset{\sim}{x},\tau)d\tau$$

$$\begin{cases} \Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t) \\ \phi^1(t) = \epsilon^{-1}\phi(t),\ \phi^n(t) = \int_{-t_h}^{t} \phi^1(t-\tau)\phi^{n-1}(\tau)d\tau,\ n \geq 2 \end{cases}$$

obtained from (3.1_1) via the usual method of successive approximations and

(iii) the elementary vector identity

$$\Delta\underline{A} = \text{grad}(\text{div } \underline{A}) - \text{curl curl } \underline{A}$$

valid for $\underline{A}(\cdot)$ sufficiently smooth on $\tilde{\Omega}$ (and applied to $\underline{E}(\cdot,t)$ at each $t \in [-t_h,\infty)$) we obtain the following equation for the evolution of the components $D_i(x,t)$ of the electric displacement field in $\tilde{\Omega} \times [-t_h,\infty)$:

$$\text{(3.2)} \qquad \frac{\partial^2 D_i}{\partial t^2}(\underline{x},t) - \frac{1}{k}\nabla^2 D_i(\underline{x},t) - \frac{1}{k}\int_{-t_h}^{t} \Phi(t-\tau)\nabla^2 D_i(\underline{x},\tau)d\tau = 0$$

where $k = \epsilon\mu$. From the homogeneity of (3.2) and the fact that $\underline{D} = \underline{0}$ in $\Omega/\tilde{\Omega}$ it follows that (3.2) is satisfied everywhere in Ω for $-t_h \leq t < \infty$. On $\partial\Omega$ we have

$$\text{(3.3a)} \qquad D_i(\underline{x},t) = 0,\ (\underline{x},t) \in \partial\Omega \times [-t_h,\infty)$$

and to this we append initial data of the form

$$\text{(3.3b)} \qquad D_i(\underline{x},0) = f_i(\underline{x}),\ \frac{\partial D_i}{\partial t}(\underline{x},0) = g_i(\underline{x}),\ x \in \bar{\Omega}$$

where it is assumed that $f_i(\cdot)$, $g_i(\cdot)$ are continuously differentiable on $\bar{\Omega}$ and vanish identically in $\Omega/\tilde{\Omega}$. In view of (3.2) and the previous specification of a past history for $\underline{E}(\underline{x},t)$ we also have

$$\text{(3.3c)} \qquad \underline{D}(\underline{x},\tau) = \begin{cases} \underline{0}, & -\infty < \tau < -t_h \\ \underline{D}_h(\underline{x},\tau), & -t_h \le \tau < 0 \end{cases}$$

for $\underline{x} \in \tilde{\Omega}$.

If we now take for H, H_+, H_- the common Sobolev spaces $H = L_2(\Omega)^3 \equiv \underline{L}_2(\Omega)$, $H_+ = (H_0^1(\Omega))^3 \equiv \underline{H}_0^1(\Omega)$, and $H_- = (H^{-1}(\Omega))^3 \equiv \underline{H}^{-1}(\Omega)$ and define operators $\underline{N} \in L_s(\underline{H}_0^1, \underline{H}^{-1})$ and $\underline{K}(\cdot) \in L^2((-\infty,\infty); L_s(\underline{H}_0^1, \underline{H}^{-1}))$ via

$$\text{(3.4a)} \qquad (\underline{N}\,\underline{v})_i = \frac{1}{k}\frac{\partial^2 v_i}{\partial x_j \partial x_j}, \quad \underline{v} \in \underline{H}_0^1(\Omega)$$

$$\text{(3.4b)} \qquad (\underline{K}(t)\underline{v})_i = -\,\phi(t)(\underline{N}\,\underline{v})_i, \quad \underline{v} \in \underline{H}_0^1(\Omega),\ t \in [-t_h,\infty)$$

then the initial-history boundary value problem (3.2), (3.3a) - (3.3c) assumes the form

$$\text{(3.5)} \qquad \begin{cases} \underline{D}_{tt} - \underline{N}\,\underline{D} + \int_{-t_h}^{t} \underline{K}(t-\tau)\underline{D}(\tau)d\tau = \underline{0}, \quad -t_h \le t < \infty \\ \underline{D}(0) = \underline{f},\ \underline{D}_t(0) = \underline{g} \\ \underline{D}(\tau) = \begin{cases} \underline{0}, & -\infty < \tau < -t_h \\ \underline{D}_h(\tau), & -t_h \le \tau < 0 \end{cases} \end{cases}$$

an initial-history value problem for $\underline{D} \in C^2([-t_h,\infty); \underline{H}_0^1)$, where we need only assume that $\underline{f}, \underline{g} \in \underline{H}_0^1(\Omega)$, $||\underline{D}_h||_{\underline{H}_0^1} \in L_1[-t_h,0)$, and where, of course, it is understood that the derivatives in (3.4a), (3.4b) are to be taken in the distribution sense.

For the spaces $\underline{L}_2(\Omega)$, $\underline{H}_0^1(\Omega)$ introduced above we have the familiar inner-products

$$(3.6)\quad \begin{cases} \langle \underline{u}, \underline{v} \rangle_{\underline{L}_2(\Omega)} = \int_\Omega u_i v_i d\underline{x} \\ \langle \underline{u}, \underline{v} \rangle_{\underline{H}_0^1(\Omega)} = \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx \end{cases}$$

The set $N'_{0,\infty}$ introduced in §2 thus has the form

$$(3.7)\quad M'_{0,\infty} = \{\underline{v} \in C([-t_h,\infty);\ \underline{H}_0^1(\Omega)) \mid \sup_{[-t_h,\infty)} [\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx]^{\frac{1}{2}} \le N_0\}$$

while

$$(3.8)\quad E(0) = \tfrac{1}{2}||\underset{\sim}{g}||^2_{\underset{\sim}{L}_2} - \tfrac{1}{2}\langle \underset{\sim}{f}, \underset{\sim}{N}\,\underset{\sim}{f}\rangle_{\underset{\sim}{L}_2}$$

$$= \tfrac{1}{2}\int_\Omega g_i g_i\, d\underset{\sim}{x} - \frac{1}{2k}\int_\Omega f_i \frac{\partial^2 f_i}{\partial x_j \partial x_j}\, dx$$

$$= \tfrac{1}{2}\int_\Omega g_i g_i d\underset{\sim}{x} - \frac{1}{2k}[\oint_{\partial\Omega} f_i \frac{\partial f_i}{\partial x_j} n_j d\underset{\sim}{x} - \int_\Omega \frac{\partial f_i}{\partial x_j}\frac{\partial f_i}{\partial x_j}\, d\underline{x}]$$

$$= \tfrac{1}{2}[\int_\Omega g_i g_i d\underset{\sim}{x} + \frac{1}{k}\int_\Omega \frac{\partial f_i}{\partial x_j}\frac{\partial f_i}{\partial x_j}\, d\underset{\sim}{x}]$$

$$= \tfrac{1}{2}[||\underline{g}||^2_{\underline{L}_2} + \frac{1}{k}||\underline{f}||^2_{\underline{H}_0^1}]$$

so that $E(0) < 0$ iff $k < 0$ with $|k| > \dfrac{||\underline{f}||^2_{H_0^1}}{||\underline{g}||^2_{\underline{L}_2}}$

Also, for any $\underline{v} \in \underline{H}_0^1(\Omega)$

$$(3.9)\quad -\langle \underline{v}, \underline{K}(0)\,\underline{v}\rangle_{\underset{\sim}{L}_2} = -\int_\Omega v_i\, [\underline{K}(0)\underline{v}]_i\, d\underline{x}$$

$$= \frac{\phi(0)}{k}\int_\Omega v_i \frac{\partial^2 v_i}{\partial x_j \partial x_j}\, d\underline{x}$$

$$= \frac{\Phi(0)}{k} \|\underline{v}\|^2_{\underline{H}^1_0} \geq 0$$

if $k < 0$ and $\Phi(0) \geq 0$. Note now that

$$\|\underline{K}(t)\|_{L_s(\underline{H}^1_0,\, \underline{H}^{-1})} = \sup_{\underline{v} \in \underline{H}^1_0} \frac{|\int_\Omega v_i\, [\underline{K}(t)\underline{v}]_i\, d\underline{x}|}{\|\underline{v}\|^2_{\underline{H}^1_0}} \tag{3.10}$$

$$= \sup_{\underline{v} \in \underline{H}^1_0} \frac{|\Phi(t)|\; |\int_\Omega v_i \frac{\partial^2 v_i}{\partial x_j \partial x_j}\, d\underline{x}|}{|k| \cdot \|\underline{v}\|^2_{\underline{H}^1_0}}$$

$$= |\Phi(t)|/|k|, \quad -\infty < t < \infty$$

and that, in a similar manner,

$$\|\underline{K}_t(t)\|_{L_s(\underline{H}^1_0,\, \underline{H}^{-1})} = |\dot{\Phi}(t)|/|k| \tag{3.11}$$

Therefore, for the problem at hand we have

$$K(t) = |\Phi(t)|/|k|, \quad K^*(t) = \frac{1}{|k|} \int |\dot{\Phi}(t)|\,dt \tag{3.12}$$

and the hypotheses (2.3) will be satisfied provided

$$\int_0^\infty |\Phi(t)|\,dt < \infty, \quad \int |\dot{\Phi}(t)|\,dt\Big|_{t=0} = 0, \text{ and}$$

$$\int_0^\infty \int_0^t |\dot{\Phi}(\lambda)|\,d\lambda\,dt < \infty.$$

With the above hypotheses relative to k, $|\Phi(\cdot)|$, $|\dot{\Phi}(\cdot)|$, $\Phi(0)$, and the additional assumption that $\langle \underline{f}, \underline{g}\rangle_{\underline{L}_2} > 0$ it then follows from the Theorem of §2 that if for $N_0 > 0$ (finite)

(3.13)
$$||\underset{\sim}{f}||^2_{\underset{\sim}{H}^1_0} \geq |k|\cdot||\underset{\sim}{g}||^2_{\underset{\sim}{L}_2} + 3\gamma N_0^2\,[\int_0^\infty |\phi(t)|dt + \int_0^\infty \int_0^t |\dot{\phi}(\lambda)|d\lambda dt]$$

there can not exist a strong solution of the initial-history value problem (3.5) which lies in the class of bounded perturbations $M'_{0,\infty}$. In passing, we remark that γ, the embedding constant for the identity map i: $\underset{\sim}{H}^1_0(\Omega) \to \underset{\sim}{L}_2(\Omega)$ depends soley on the region Ω. For Ω such that γ is very small ($<\delta_1$) and with a memory function $\phi(t)$ such that $||K||_{L_1[0,\infty)}$ and $||K^*||_{L_1[0,\infty)}$ are also very small ($<\delta_2$'6) (3.13) is implied by

(3.14)
$$||\underset{\sim}{f}||^2_{\underset{\sim}{H}^1_0} \geq |k|\cdot||\underset{\sim}{g}||^2_{\underset{\sim}{L}_2} + \delta_1\delta_2 N_0^2$$

which indicates that as $\delta_1\delta_2 \to 0$ the existence in $M'_{0,\infty}$ of a solution $\underset{\sim}{u}$ to the initial-history value problem (3.5) depends essentially on the relationship between the relative magnitudes in $\underset{\sim}{H}^1_0$ and $\underset{\sim}{L}_2$ of the initial-values of the electric displacement field and its first time derivative.

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